Prices of Uncertainty and Long-Run Risk

Summer Camp in Applied Econometrics
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2018/08/12

Outline

- Empirical Asset Pricing
- Equity Premium Puzzle
- Consumption Based Asset Pricing Model with Recursive Utility
- Measuring Long-Run Risk and Price of Long-Run Risk
- Long-Run Risk vs Uncertainty

Empirical Asset Pricing

- Main question: What risk are priced in the market? Is the market efficient?
- Main test: Linear Factor Model (CAPM)

$$
\begin{aligned}
E_t[R_{i,t+1} - R_{f,t}] &= \beta_{i,t} \cdot \lambda_t \text{ (conditional)} \\
E[R_{i,t+1} - R_{f,t}] &= \beta_i \cdot \lambda \text{ (unconditional)}
\end{aligned}
$$

where

$$
\begin{aligned}
\beta_i &: \text{ risk exposure} \\
\lambda &: \text{ price of risk}
\end{aligned}
$$

- Regression

$$R_{i,t+1} - R_{f,t} = \alpha_i + \beta_i \cdot \lambda_t + \varepsilon_{i,t+1}$$

$$H_0 : \alpha_i = 0$$

Empirical Asset Pricing

- The underlying assuptions:
  - The Market is efficient
  - The model captures all the risk factors that are priced in the market
    - How to measure risk?
- Note:
  - "Joint-Test Problem" in Empirical Tests of the EMH: Market Efficiency per se is not testable
    - The question whether price reflects a given piece of information always depends on the model of asset pricing that the researcher is using.
    - It is always a joint test of market efficiency and the used pricing model.
  - Despite the joint-test problem, tests of market efficiency, i.e. search for anomalies or "arbitrage" opportunities, improves our understanding of the behavior of returns across time and securities. It helps to improve existing asset pricing models and understanding of financial markets.

Equity Premium Puzzle

- Consumption-based asset pricing model with CRRA Utility:

$$E_t \sum_{j=0}^{\infty} \beta^j u(c_{t+j})$$

Euler Equation:

$$E_t \left[\beta \left(\frac{u'(c_{t+1})}{u'(c_t)} \right) R^i_{t+1} \right] = 1$$

which implies that

$$\begin{aligned} E_t \left[R^{ei}_{t+1} \right] &= -\frac{1}{R^f_t} cov_t \left(R^e_{t+1}, \beta \frac{u'(c_{t+1})}{u'(c_t)} \right) \\ \frac{1}{R^f_t} &= E_t \left[\beta \left(\frac{u'(c_{t+1})}{u'(c_t)} \right) \right] \end{aligned}$$

Equity Premium Puzzle

- With CRRA (power) utility

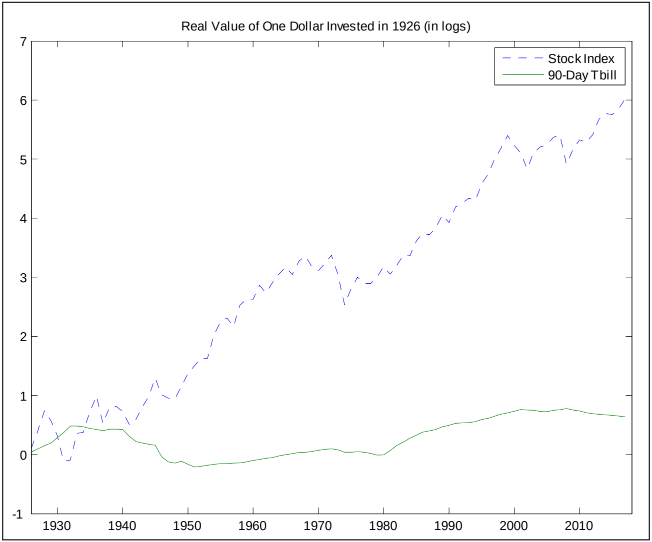
$$u'(c) = c^{-\gamma}$$

We have

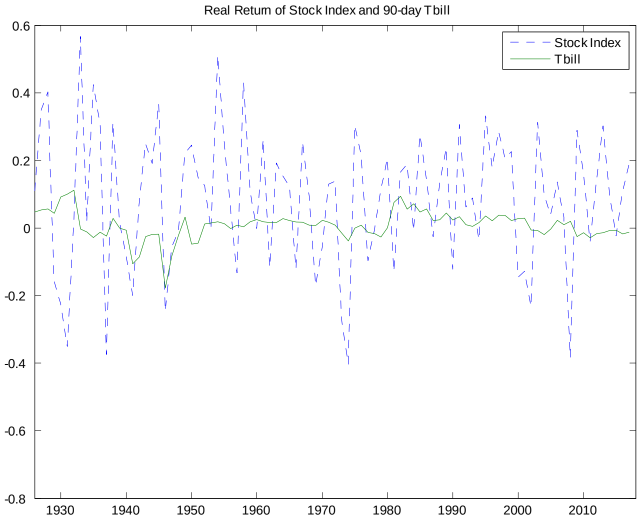
$$\begin{aligned} E_t\left[R^{ei}_{t+1}\right] &\propto \gamma cov_t\left(R^{e}_{t+1}, \frac{c_{t+1}}{c_t}\right) \\ &= \gamma\sigma_t(R^{ei}_{t+1})\sigma_t(\Delta c_{t+1})\rho_t(\Delta c_{t+1}, R^{ei}_{t+1}) \\ &\Longrightarrow \left|\frac{E_t\left[R^{ei}_{t+1}\right]}{\sigma_t(R^{ei}_{t+1})}\right| < \gamma\sigma_t(\Delta c_{t+1}) \end{aligned}$$

- In postwar U.S. data, the mean return of stocks over bonds is about 5 percent, with a standard deviation of about 20 percent, so the Sharpe ratio
is about 0.25. Aggregate non-durable and services consumption volatility is much smaller, about 1 percent per year. We need a risk aversion of at least 25!

Equity Premium Puzzle



Equity Premium Puzzle



Equity Premium Puzzle

Table: Summary Statistics of Real Return

(%)	R^e	Rf90	Rf30	ΔP (CPI)	
	Panel A: 1926-2017				
Annual	8.81	0.78	0.42	2.97	
std dev	20.31	4.12	3.93	4.04	
Quarterly	2.66	0.19	0.11	0.72	
std dev	17.45	1.32	1.28	1.31	
Monthly	0.75	0.06	0.04	0.24	
std dev	7.21	0.54	0.53	0.53	
	Panel B: 1947-2017 Annualized				
	R^e	Rf90	Rf30	ΔP (PCE)	Δc_t
Quarterly	*6.34*	*1.25*	0.83	3.14	*3.21*
std dev	*21.67*	*1.58*	1.45	1.47	*1.09*
corr with Δc_t	*0.19*	0.26	0.30	-0.28	1

Equity Premium Puzzle

- Initial Reaction
  1. T-bill rate is not a good approximation for the risk-free interest rate?
     No, as high sample Sharpe ratios are pervasive in finance and not limited to the difference between stocks and bonds
  2. Risk aversion is indeed high?
     No, the implied risk free rate is more than 20%. Assuming log-normal distribution

$$r_t^f = \log \beta + \gamma E_t(\Delta c_{t+1}) - \frac{1}{2}\gamma^2 \sigma_t^2(\Delta c_{t+1})$$

     –> Risk Free Rate Puzzle
  3. More information is worse, $\rho = 0.2$, implies risk aversion of more than 100!

Equity Premium Puzzle

- Equity Premium and Risk Free Rate Puzzle
  1. Quantitative not qualitative puzzle,
  2. Consumption is proportional to wealth in the derivation of the CAPM, so the CAPM predicts that consumption should inherit the large 20 percent or so volatility of the stock market
  3. Implication optimal portfolio choice:

     $$w = \frac{1}{\gamma}\frac{E(R^e)}{\sigma^2(R^e)}$$

     100% equity investment for γ around 3. However, conditonal on 20% consumption std.dev.!!
  4. Consumption is much smoother that wealth, but consumption-wealth ratio is stationary in the long-run
     $\Longrightarrow$ Lettau and Ludvigon's *cay* as a forecasting variable.

Equity Premium Puzzle

- Empirical Tests:
  1. Mehra and Prescott (1985): Calibration
  2. Hansen and Singleton (1983): GMM of conditional and unconditional Euler equation

$$
\begin{aligned}
E_t\left[\beta\left(\frac{u'(c_{t+1})}{u'(c_t)}\right)R^i_{t+1}\right] &= 0 \\
E\left\{\left[\beta\left(\frac{u'(c_{t+1})}{u'(c_t)}\right)R^i_{t+1}-1\right]z_t)\right\} &= 0
\end{aligned}
$$

where z_t consists of lags of consumption and returns, which do not forecast either consumption growth or returns very well.

Equity Premium Puzzle

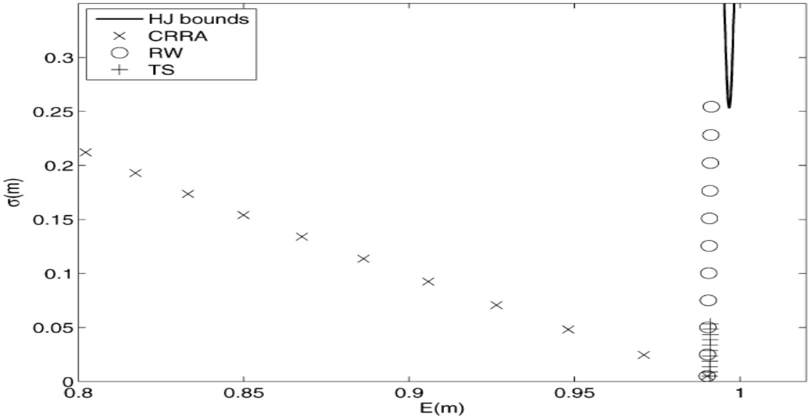


Figure: Barillas et al.(2009) Hansen–Jagannathan volatility bound for quarterly returns on the value-weighted NYSE and Treasury Bill, 1948–2006.

Equity Premium Puzzle

- Risk Aversion and Intertemporal Substitution
  1. Risk aversion: measures the risk attitude
  2. Intertemporal elasticity of substitution: how much consumption growth changes when interest rates go up 1 percent, measures attitude towards behavior of a single asset over time and in particular to line up variation in expected consumption growth with variation in risk-free interest rates
  3. Inverse to each other in CRRA utility
  4. Quite high risk aversion is required to digest the equity premium is robust in consumption-based model estimation
  5. Much more debate on IES
     - Hansen and Singleton found numbers near one
     - Hall (1988) argued the estimate should be closer to zero
     - Campbell (2003) argues for small IES, as we observe small movements in expected consumption growth against large movements in real interest rates

Equity Premium Puzzle

- Questions:
  1. What utility function should one use?
  2. How should one treat time aggregation and consumption data?
  3. How about multiple goods?
  4. What asset returns and instruments are informative?
  5. Asset pricing empirical work has moved from industry or beta portfolios, the use of lagged returns, and consumption growth as instruments to the use of size, book-to-market, momentum portfolios, and the dividend-price ratio, term spreads, and other more powerful instruments.

     How does the consumption-based model fare against this higher bar?
  6. The data may be poor enough that practitioners will still choose "reduced-form" financial models, but economic understanding of the stock market must be based on the idea that people fear stocks, and hence do not buy more despite attractive returns, because people fear that stocks will fall in "bad times." At some point "bad times" must be mirrored in a decision to cut back on consumption.

- Later Responses:
  - Non-separable utility across goods: leisure (Eichenbaum, Hansen, and Singleton (1988)), durable goods (Yogo(2004), Pakos (2004))
  - Non-separable utility over time goods: habit (Constantinides (1991), Abel (1990), Heaton (1993, 1995) Campbell and Cochrane (2000))

$$U = \sum_t \beta^t u(k_t) = \sum_t \beta^t \left(\sum_{j=0}^{\infty} (1-\delta)^j c_{t+j} \right)$$
$$k_{t+1} = (1-\delta) k_t + c_{t+1}$$

or

$$U = \sum_t \beta^t u(c_t - \theta x_t), x_t = \rho x_{t-1} + \lambda c_t$$

SDF:

$$M_{t+1} = \beta \left(\frac{c_{t+1}}{c_t} \right)^{-\gamma} \left(\frac{s_{t+1}}{s_t} \right)^{-\gamma} . s_t = \frac{c_t - x_t}{c_t}$$

Equity Premium Puzzle

- Non-separable utility across states of nature:
  - Epstein and Zin (1989) recursive utility

$$U_t = \left((1-\beta) c_t^{1-\rho} + \beta \left(E_t \left(U_{t+1}^{1-\gamma} \right) \right)^{\frac{1-\rho}{1-\gamma}} \right)^{1/(1-\rho)}$$

  - SDF

$$M_{t+1} = \beta \left(\frac{c_{t+1}}{c_t} \right)^{-\rho} \left(\frac{U_{t+1}}{E_t \left(U_{t+1}^{1-\gamma} \right)^{1/(1-\rho)}} \right)^{\rho-\gamma}$$

  - Challenges: how to compute $E_t \left(U_{t+1}^{1-\gamma} \right)$?

Equity Premium Puzzle

- Two ways:
  - Campbell (1996), Bansal and Yaron (2004) and etc: Use return on wealth to proxy and log-linearize around stationary consumption-wealth ratio

    $$m_{t+1} = a - bR_{t+1}^{W}$$

    Bansal, Dittmar, and Lundblad (2005) also argue that average returns of value vs. growth stocks can be understood by different covariances with long-run consumption growth in this framework
  - Hansen, Heaton, and Li (2008): Log-linearize around $\rho = 1$.

    $$\begin{aligned}(E_{t+1} - E_t)m_{t+1} &= -\gamma(E_{t+1} - E_t)\Delta c_{t+1} \\ &\quad +(1-\gamma)(E_{t+1} - E_t)\left[\sum_{j=1}^{\infty} \beta^j (\Delta c_{t+1+j})\right]\end{aligned}$$

  - Parker and Julliard (2005): the return at $t+1$ predicts a string of small changes in consumption growth Δc_{t+j}
  - Piazzesi and Schneider (2006) apply the framework to bonds. They generate risk premia in the term structure by the ability of state variables to forecast future consumption growth.

Equity Premium Puzzle

- Questions:
    - Is the elasticity of intertemporal substitution really that different from the coefficient of risk aversion?
    - Are there really important dynamics in consumption growth?
    - Hansen, Heaton, and Li (2008) sensitivity analysis show that long-run properties of anything are hard to measure, Bansal, Dittmar, and Lundblad's finding of a strong beta to explain value premia depends crucially on the inclusion of a time trend in the regression of earnings on consumption.
    - Large risk aversion is still needed.

Asset Pricing Under Recursive Utility

- Recursive utility:

$$V_t = \{(1-\beta)C_t^{1-\rho} + \beta[\mathcal{R}_t(V_{t+1})]^{1-\rho}\}^{\frac{1}{1-\rho}}$$

  where V_{t+1} is the continuation value of a consumption plan from time $t+1$, $\mathcal{R}_t(V_{t+1}) = [E_t(V_{t+1}^{1-\gamma})]^{\frac{1}{1-\gamma}}$ is *certainty equivalent* of random future utility governed by risk preferences $\gamma > 0$ is RRA for static wealth gamble; $1/\rho > 0$ is the IES

- Budget Constraint:

$$W_{t+1} = R_{t+1}^W (W_t - C_t)$$

- This recursion is homogeneous of degree 1 in W_t, hence when consumption is chosen optimally,

$$V_t^* = \left[(1-\beta)^{-1/\rho}\frac{C_t^*}{W_t}\right]^{1/(1-1/\rho)} W_t$$

  Note: This solution is not the full solution

Asset Pricing Under Recursive Utility

- This recursion is homogeneous of degree 1 in (C_t, V_{t+1}) hence

$$\begin{aligned} V_t &= (MC_t) C_t + E_t (MV_{t+1} V_{t+1}) \\ MC_t &= \frac{\partial V_t}{\partial C_t} = (1-\beta) V_t^{\rho} C_t^{-\rho} \\ MV_{t+1} &= \frac{\partial V_t}{\partial V_{t+1}} = \beta \left(\frac{V_t}{R_t(V_{t+1})} \right)^{\rho} \left(\frac{V_{t+1}}{R_t(V_{t+1})} \right)^{-\gamma} \end{aligned}$$

- Stochastic discount factor:

$$S_{t+1} = \frac{MV_{t+1} MC_{t+1}}{MC_t} = \beta \left(\frac{C_{t+1}}{C_t} \right)^{-\rho} \left(\frac{V_{t+1}}{\mathcal{R}_t(V_{t+1})} \right)^{\rho-\gamma} \qquad (1)$$

- Euler Equation:

$$E_t \left[S_{t+1} R_{t+1} \right] = 1$$

Asset Pricing Under Recursive Utility

- Problem: How to measure V_{t+1} or W_{t+1}
  - Epstein and Zin (1991) and Campbell (1996) use the link between return on wealth and continuation value.
  - Give a well-specified stochastic process governing consumption and avoid the need to construct a proxy to the return on wealth.
    - Restoy and Weil (1998) and Bansal and Yaron (2004): Loglinearize around a constant consumption-wealth ratio
    - Hansen, Heaton and Li (2008): linearize around $\rho = 1$.

Asset Pricing Under Recursive Utility

- Return on wealth portfolio:

$$R^w_{t+1} = \frac{W_{t+1}}{W_t - C_t}$$

$\Longrightarrow$

$$R^W_{t+1} = \frac{1}{\beta}\left(\frac{C_{t+1}}{C_t}\right)^{\rho}\left(\frac{V_{t+1}}{\mathcal{R}_t(V_{t+1})}\right)^{1-\rho} \tag{2}$$

from (2),we have the continuation value could be written in term of R^W_{t+1} and C_{t+1}/C_t,

$$(\frac{V_{t+1}}{\mathcal{R}_t(V_{t+1})})^{1-\rho} = \beta\left(\frac{C_{t+1}}{C_t}\right)^{-\rho} R^W_{t+1}$$

from (1)and (2), we have the SDF written in term of R^W_{t+1} and C_{t+1}/C_t

$$S_{t+1} = \left[\beta\left(\frac{C_{t+1}}{C_t}\right)^{-\rho}\right]^{\frac{1-\gamma}{1-\rho}}\left(R^W_{t+1}\right)^{\frac{\rho-\gamma}{1-\rho}}$$

Asset Pricing Under Recursive Utility

- Euler equation for wealth portfolio,

$$E_t\left[\left[\beta\left(\frac{C_{t+1}}{C_t}\right)^{-\rho}R^W_{t+1}\right]^{\frac{1-\gamma}{1-\rho}}\right]=1 \tag{3}$$

- Plug in any asset return R^i_{t+1} in the Euler equation we have

$$\begin{aligned}E_t r^i_{t+1}-r^f_t+\frac{1}{2}var_t(r^i_{t+1}) \quad = \quad & (1-\theta)cov_t(r^w_{t+1},r^i_{t+1}) \\ & +\theta\rho cov_t(r^i_{t+1},\Delta c_{t+1})\end{aligned}$$

Special cases:

- $\theta=1$, i.e. $\rho=\gamma$, power utility
- $\theta=0$,i.e. $\gamma=1$, *static* CAPM pricing formula
- $\rho=1$ then $\frac{C_t}{W_t}$ is constant.

Asset Pricing Under Recursive Utility

- How to measure r^w?
  - Epstein and Zin (1991): use stock market return as proxy
  - Campbell (1996): include human capital return, which has assigned market or shadow value, assume the share of financial wealth and labor income are stable
    and obtain a three-factor model:

$$r^w_{t+1} = (1-\nu)r^m_{t+1} + \nu r^y_{t+1} + \hat{k}$$

  - Santos and Veronesi: shares varies over time, which is an important explanatory variable $(\nu- > \nu_t)$
  - Letteau and Ludvigson (2001): cointegration relationship is important. (cay)

Hansen, Heaton and Li (2008)

- Under the assumption of linear dynamics of consumption and dividend growth, we explicitly solve the value function and stochastic discount factor when $\rho = 1$. In the more general case of $\rho \neq 1$, we approximate the solution around $\rho = 1$.
- Solve the model for $\rho = 1$
  - Take logs of value function

$$\begin{aligned} v_t &\equiv \log(\frac{V_t}{C_t}) \\ &= \frac{1}{1-\rho} \log\left((1-\beta) + \beta \exp\left((1-\rho) Q_t\left(v_{t+1} + \Delta c_{t+1}\right)\right)\right) \end{aligned}$$

where

$$\begin{aligned} Q_t\left(v_{t+1} + \Delta c_{t+1}\right) &\equiv \log \mathcal{R}_t\left(\frac{V_{t+1}}{C_{t+1}} \frac{C_{t+1}}{C_t}\right) \\ &= \frac{1}{1-\gamma} \log E_t\left[\exp((1-\gamma)\left(v_{t+1} + \Delta c_{t+1}\right))\right] \end{aligned}$$

  - Stochastic Discount Factor: $s_{t+1} \equiv \log S_{t+1}$

Solve the model for IES =1

- Specification of dynamics: state vector x_t follows VAR, dividend growth and consumption growth is linear in states.

$$
\begin{aligned}
x_{t+1} &= Gx_t + Hw_{t+1} \\
\Delta c_{t+1} &= \mu_c + U_c \cdot x_t + \lambda_0 \cdot w_{t+1} \\
d_{t+1} - d_t &= \mu_d + U_d \cdot x_t + \iota_0 \cdot w_{t+1}
\end{aligned} \tag{4}
$$

where $w_{t+1} \tilde{} N(0, I)$ is a i.i.d normally distributed vector.

Solve the model for IES =1

- We are interested in the discounted response of consumption growth and dividend growth to the shocks, namely, $\lambda(\beta)$ and $\iota(\beta)$ defined as following

$$
\begin{aligned}
\lambda(\beta) &= \sum_{j=0}^{\infty} \beta^j \frac{\partial \Delta c_{t+1+j}}{\partial w_{t+1}} = \lambda_0 + U_c \sum_{j=1}^{\infty} \beta^j G^{j-1} H \quad (5) \\
&= \lambda_0 + \beta U_c (I - \beta G)^{-1} H \\
\lambda(1) &= \lambda_0 + U_c (I - G)^{-1} H \quad (6) \\
\iota(\beta) &= \sum_{j=0}^{\infty} \beta^j \frac{\partial \Delta d_{t+1+j}}{\partial w_{t+1}} = \iota_0 + U_d \sum_{j=1}^{\infty} \beta^j G^{j-1} H \quad (7) \\
&= \iota_0 + \beta U_d (I - \beta G)^{-1} H \\
\iota(1) &= \iota_0 + U_d (I - G)^{-1} H \quad (8)
\end{aligned}
$$

Solve the model for IES =1

- Dividend dynamics is given as

$$D_t = D_t^* f(x_t), D_t^* = \exp\left[\zeta t + \sum_{j=1}^{t} \pi w_j\right]$$

where D_t^* is a geometric random walk (governs long-run growth rate) and $f(x_t)$ is the transitory component, let $d_t = \log(D_t)$

$$d_{t+1} - d_t = \zeta + \pi w_{t+1} + \log f(x_{t+1}) - \log f(x_t) \tag{9}$$

and we need to link it to the VAR system (4). We do this by performing martingale extraction

$$d_{t+1} - d_t = \mu_d + U_d x_t + \iota_0 w_{t+1} = \mu_d + \iota(1) w_{t+1} - U_d^*(x_{t+1} - x_t)$$

where

$$\iota(1) = \iota_0 + U_d(I - G)^{-1} H, U_d^* \equiv U_d(I - G)^{-1}$$

Solve the model for IES $=1$

- When $\rho = 1$, the recursion of value function and SDF became

$$\begin{aligned} v_t &\equiv \log V_t - \log C_t = \frac{\beta}{1-\gamma} \log E_t[\exp(1-\gamma)(v_{t+1} + \Delta c_{t+1})] \\ &\equiv \beta Q_t(v_{t+1} + \Delta c_{t+1}) \\ s_{t+1} &= \log \beta - \Delta c_{t+1} + (1-\gamma)[v_{t+1} + \Delta c_{t+1} - Q_t\left(v_{t+1} + \Delta c_{t+1}\right)] \end{aligned}$$

- Guess and Verify: value function and SDF is log-linear in the states

Solve the model for IES =1

- Value function

$$v_t = \mu_v + U_v x_t$$
$$U_v = \beta U_c (I - \beta G)^{-1}$$
$$\mu_v = \frac{\beta}{1-\beta}\left[\mu_c + \frac{(1-\gamma)|\lambda(\beta)|^2}{2}\right]$$

- SDF

$$s_{t+1,t} = \mu_s + U_s x_t + \xi_0 \varepsilon_{t+1}$$
$$\mu_s \equiv \log\beta - \mu_c - \frac{(1-\gamma)^2|\lambda(\beta)|^2}{2}$$
$$U_s \equiv -U_c$$
$$\xi_0 = (1-\gamma)\lambda(\beta) - \lambda_0$$

Solve the model for IES =1

- Long-Run Pricing Operator and Growth Operator
  Given the dynamics of dividends, the pricing vector is obtained by finding the dominant eigenvalue and eigenvector of the pricing operator and growth operator defined as following:

$$\frac{P_t}{D_t^*} = E\left[\exp(s_{t+1,t})\frac{D_{t+1}}{D_t^*}|x_t\right] \tag{10}$$
$$= E\left[\exp(s_{t+1,t} + \zeta + \pi w_{t+1}) f(x_{t+1})|x_t\right] \equiv \mathcal{P} f(x_t)$$
$$E_t\left[\frac{D_{t+1}}{D_t^*}|x_t\right] = E\left[\exp(\zeta + \pi w_{t+1}) f(x_{t+1})|x_t\right] \equiv \mathcal{G} f(x_t) \tag{11}$$

- We can show that for any function $f(x)$ that is log-linear in the Markov state x, the function $\mathcal{P}f$ and $\mathcal{G}f(x_t)$is also log-linear in the Markov state x, that is,

$$\mathcal{P}\exp(-\omega x + \kappa) = \exp(-\omega' x + \kappa')$$
$$\mathcal{G}\exp(-\omega^g x + \kappa^g) = \exp(-\omega^{g\prime} x + \kappa^{g\prime})$$

Solve the model for IES =1

- $\mathcal{P}^j f$ is also log-linear in x and satisfies the recursion

$$\omega^{(j)} = \omega^{(j-1)} G - U_s \tag{12}$$

$$\kappa^{(j)} = \kappa^{(j-1)} + \mu_s + \zeta + \frac{|\xi_0 + \pi - \omega^{(j-1)} H|^2}{2}$$

with

$$\omega^0 = \omega, \kappa^0 = \kappa$$

As $j \longrightarrow \infty$, the limit of $\omega^{(j)}$ is the fixed point of (12), and

$$\bar{\omega} = \lim_{j\to\infty} \omega^{(j)} = -U_s (I - G)^{-1} = U_c \left(I - G\right)^{-1} \tag{13}$$

$$-\nu = \lim_{j\to\infty} \left[\kappa^{(j)} - \kappa^{(j-1)}\right] = \mu_s + \zeta + \frac{|\xi_0 + \pi - \bar{\omega} H|^2}{2} \tag{14}$$

- The price of a claim to a single cash flow $D_{t+j} = D^*_{t+j} f\left(x_{t+j}\right)$ decays at a rate which is asymptotically constant (ν).

Solve the model for IES =1

- Plug in the solution of SDF and dividend dynamics we have[1]

$$-\nu = \log\beta - \mu_c + \mu_d + \frac{|\iota(1) - \lambda(1)|^2}{2} + (1-\gamma)\lambda(\beta)\cdot[\iota(1) - \lambda(1)]$$

In fact $-\nu$ and $\exp(\bar{\omega}x)$ are the dominant eigenvalue and eigenfunction of pricing operator, that is

$$\mathcal{P}\exp(\bar{\omega}x) = \exp(-\nu)\exp(\bar{\omega}x)$$

- Similarly, the dominant eigenvalue and eigenfunction of growth operator is the fixed point of the growth rate recursion

$$\begin{aligned} \mathcal{G}\exp(\bar{\omega}^g x) &= \exp(\eta)\exp(\bar{\omega}^g x) \\ \bar{\omega}^g &= 0, i.e. \exp(\bar{\omega}^g x) = 1 \\ \eta &= \zeta + \frac{|\pi|^2}{2} = \mu_d + \frac{|\iota(1)|^2}{2} \end{aligned}$$

- As j gets large, we can show that for any $f(x)$

$$\mathcal{P}^j f \simeq \exp(-j\nu)\exp(\bar{\omega}x), \;\; \mathcal{G}^j f \simeq \exp(j\eta)\exp(\bar{\omega}^g x) = \exp(j\eta)$$

Solve the model for IES $=1$

- Long-Run Return: Consider the expected return to holding a claim to a single cash flow D_{t+j}

$$\begin{aligned} E_t(R_{t,t+j}) &= E_t\left[\frac{D_{t+j}}{P_t}\right] = \frac{E_t\left[D_{t+j}/D_t^*\right]}{P_t/D_t^*} \\ &= \frac{\mathcal{G}^j f}{\mathcal{P}^j f} = ((\eta+\nu)j)\exp(-\bar{\omega}x) \end{aligned}$$

- Hence the APR as j gets large is then

$$\lim_{j\longrightarrow\infty}\frac{1}{j}\log E_t(R_{t,t+j}) = \eta+\nu$$

- The long-run rate of return can be rewrite as

$$\nu+\eta = \underset{\text{(long-run risk free rate)}}{\varsigma^*} + \underset{\text{(long-run price of risk)}}{\pi^*} \cdot \underset{\text{(long-run risk exposure)}}{\pi}$$

$$\pi^* = \lambda(1) - (1-\gamma)\lambda(\beta)$$

$$\varsigma^* = -\log\beta + \mu_c - \frac{|\lambda(1)|^2}{2} - (1-\gamma)\lambda(\beta)\cdot\lambda(1)$$

Solve the model for general IES

- In this case we find approximating solution around $\rho = 1$.

$$
\begin{aligned}
s_{t+1,t} &= s^1_{t+1,t} + (\rho - 1) Ds^1_{t+1,t} \\
Ds^1_{t+1,t} &= \frac{1}{2} w'_{t+1} \Theta_0 w_{t+1} + w'_{t+1} \Theta_1 x_t + \vartheta_0 + \vartheta_1 x_1 + \vartheta_2 w_{t+1}
\end{aligned}
$$

- Long-run Return

$$
\eta + \nu = \varsigma^* + \pi \pi^* + \frac{1}{2} \pi \Pi^* \pi'
$$

In this case, the return not only depends linearly on the risk of the cash flows (π) but also depends on the quadratic of the risk of the cash flows.

Estimating Long-Run Price of Risk

- First estimate VAR

$$
\begin{aligned}
x_{t+1} &= Gx_t + Hw_{t+1} \\
\Delta c_{t+1} &= \mu_c + U_c \cdot x_t + \lambda_0 \cdot w_{t+1} \\
d_{t+1} - d_t &= \mu_d + U_d \cdot x_t + \iota_0 \cdot w_{t+1}
\end{aligned} \tag{15}
$$

  We impose the dividends growth does not Granger cause the state variables x.

- Note: Cointegration test is very sensitive to the model specification (lags, with or without trend)

Estimating Long-Run Price of Risk

- Next, find suitable state variables to identify important shocks that matters for the long-run:
  - In Hansen, Heaton and Li (2008): aggregate consumption and earnings (corporate profit) are cointegrated $\Longrightarrow$ one permanent shock

$$y_t = \begin{bmatrix} c_t - c_{t-1} \\ e_t - c_t \end{bmatrix}$$

$$y_t = \mu_y + B_1 y_{t-1} + ... + B_l y_{t-l} + V\varepsilon_t$$

  - In Li (2018): aggregate investment, consumption and relative price of investment goods, with nominal consumption and investment are cointegrated $\Longrightarrow$ two permanent shock, one aggregate tech shock, one investment-specific technological shock

$$y_t = \begin{bmatrix} i_t - i_{t-1} \\ c_t - c_{t-1} \\ i_t + pi_t - c_t \end{bmatrix}$$

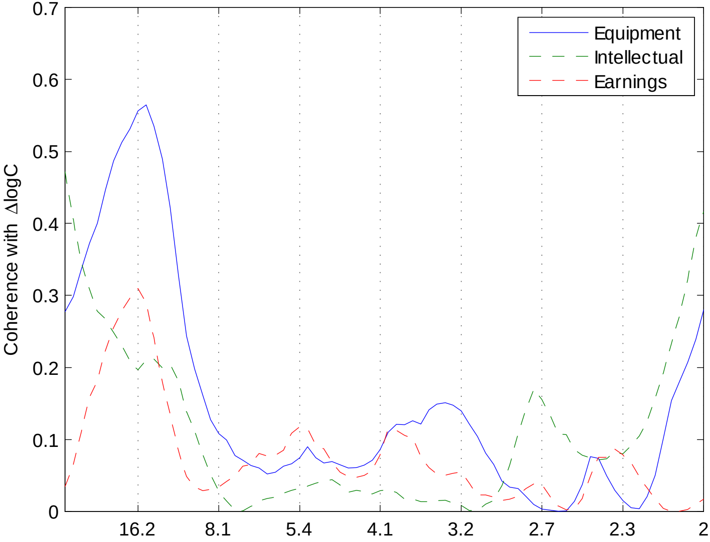
$$y_t = \mu_y + B_1 y_{t-1} + ... + B_l y_{t-l} + V\varepsilon_t$$

- Identification of shocks: Sims (1972) or Blanchard and Quah (1989)

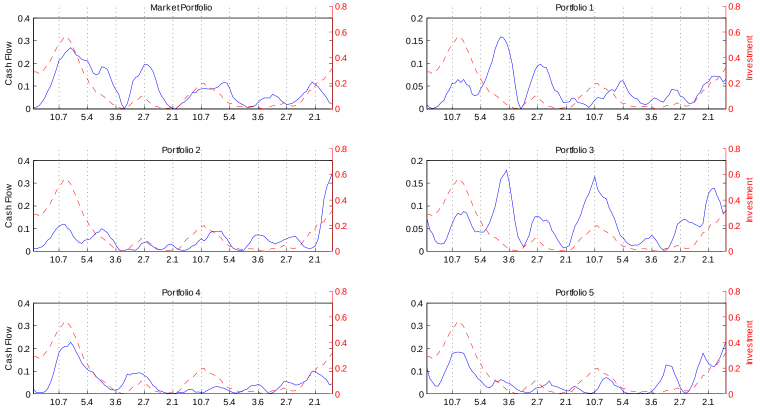
R&D Portfolio

EW	mean ret	std. dev	mean grwoth	std. dev
crspmkt	1.23%	5.28%	2.46%	1.38%
mkt	1.47%	6.67%	2.59%	2.46%
RDNA	1.22%	5.56%	2.27%	2.28%
Rdzero	1.23%	5.88%	2.49%	4.09%
1	1.32%	5.38%	2.42%	1.80%
2	1.39%	5.53%	2.46%	2.25%
3	1.58%	6.65%	2.78%	4.13%
4	1.67%	7.83%	2.66%	7.08%
5	1.39%	9.58%	2.12%	14.47%

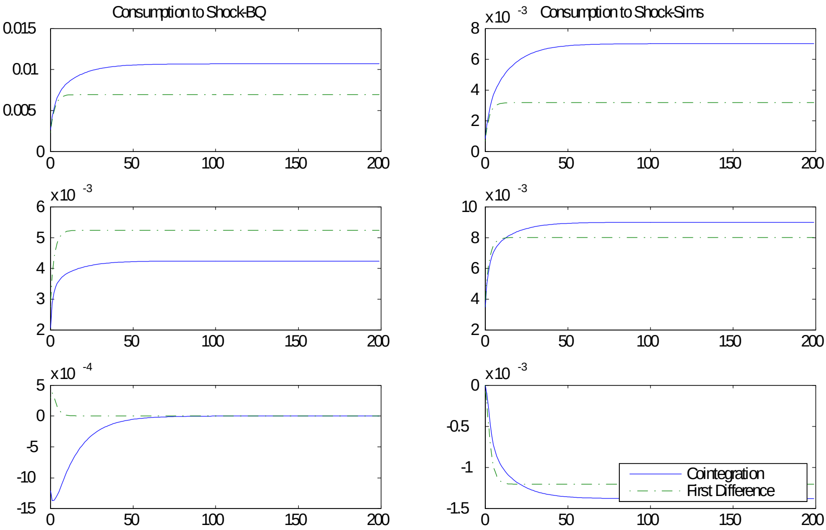
Cohenrence with Consumption Growth



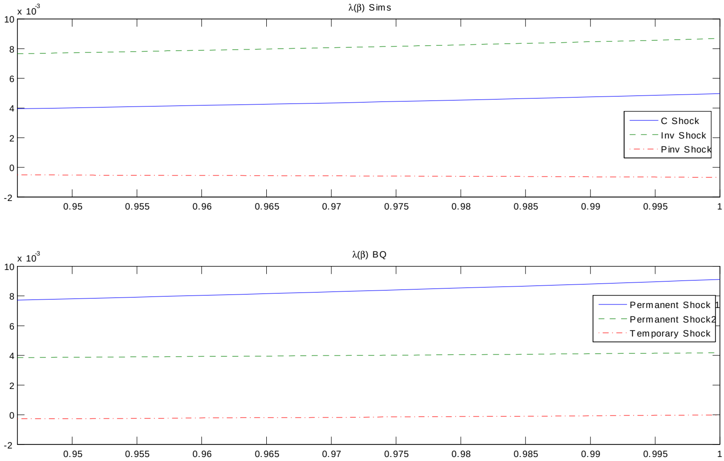
Cohenrence with Consumption Growth



Impulse Responses to Shocks



Long-Run Price of Risk



Price of Long-Run Risk or Price of Uncertainty

- In the long-run risk model, a large risk aversion is still needed to generate sizable risk premium
  - No more risk free rate puzzle
  - But as Lucas said
    "No one has found risk aversion parameters of 50 or 100 in the diversification of individual portfolios, in the level of insurance deductibles, in the wage premiums associated with occupations with high earnings risk, or in the revenues raised by state-operated lotteries. It would be good to have the equity premium resolved, but I think we need to look beyond high estimates of risk aversion to do it."

Price of Long-Run Risk or Price of Uncertainty

- Barillas, Hansen, and Sargent (2009) show that a max–min expected utility theory lets us reinterpret risk-aversion parameter in EZ preference when IES $=1$ as measuring a representative consumer's doubts about the model specification. Hence, reinterprete the (log-run) market price of risk as a price of model uncertainty.
  - Doubts about the model specification can be substantial
  - Prices of model uncertainty contains information about the benefits of removing model uncertainty, instead of the "consumption risk" that Lucas studied.
  - Allow us to estimate the welfare cost of increasing model uncertainty due to policy uncertainty

Price of Long-Run Risk or Price of Uncertainty

- Agent with ambiguity aversion

$$\begin{aligned} W(x_0) &= \min_{[g_{t+1}]} \sum_{t=0}^{\infty} E_0 \left[\beta^t G_t \left(c_t + \beta\theta E_t(g_{t+1} \log(g_{t+1}))\right)\right] \\ s.t. G_{t+1} &= g_{t+1} G_t,\, E_t[g_{t+1}] = 1,\, g_{t+1} \geq 0 \\ x_{t+1} &= Ax_t + B\varepsilon_{t+1} \\ c_t &= Hx_t \end{aligned}$$

where θ indicates fear of model misspecification as measured by how much the minimizing agent gets penalized for raising entropy.

- Observationally equivalent to V for $\rho = 1$ with

$$\theta = \frac{1}{(1-\beta)(1-\gamma)}$$

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